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Magnetic monopoles in ferromagnetic spin-triplet superconductors

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Abstract
Using the ϕ-mapping method, we argue that ferromagnetic spin-triplet superconductors allow the formation of unstable magnetic monopoles. In particular, we show that the limit points and the bifurcation points of the ϕ-mapping will serve as the interaction points of these magnetic monopoles.

Topological solitons play important roles in many fields of physical science ranging from condensed matter physics to QCD [1]. In [2], Babaev derived a dual presentation of the free energy for ferromagnetic spin-triplet superconductors in terms of gauge invariant variables. The similarity between this dual presentation and the energy function of the Faddeev model [3] reveals the nontrivial topological structure of these superconductors. Based on this topological structure, one can conclude that these superconductors allow the formation of stable knotted solitons [2]. In this paper, making use of the ϕ-mapping method [4], we argue that ferromagnetic spin-triplet superconductors allow the formation of unstable magnetic monopoles. In particular, we show that at the limit points of the ϕ-mapping, these magnetic monopoles will be created or annihilated in pairs, and at the bifurcation points of the ϕ-mapping, they will interact with each other.

First, we review the dual presentation of the free energy for ferromagnetic spin-triplet superconductors. We write the order parameter of the spin-triplet Bose condensate as $\Psi(\mathbf{x},t) = \sqrt{n}(\mathbf{x},t)\zeta(\mathbf{x},t)$, where n is the total density and ζ is a normalized spinor. Then the free energy of the spin-triplet superconductor reads [2]

$$F = \int \mathrm{d}\mathbf{x}\left[\frac{\hbar^2}{2M}(\nabla\sqrt{n})^2 + \frac{\hbar^2 n}{2M}\left|\left(\nabla + i\frac{2e}{\hbar c}\mathbf{A}\right)\zeta\right|^2 - \mu n + \frac{n^2}{2}\left[c_0 + c_2\langle\mathbf{F}\rangle^2\right] + \frac{\mathbf{B}^2}{8\pi}\right], \quad (1)$$

where the average spin $\langle\mathbf{F}\rangle = \zeta^\dagger\mathbf{F}\zeta$. All degenerate spinors are related to each other by gauge transformation $\mathrm{e}^{\mathrm{i}\theta}$ and spin rotations $\mathcal{U}(\alpha,\beta,\tau) = \mathrm{e}^{-\mathrm{i}F_z\alpha}\mathrm{e}^{-\mathrm{i}F_y\beta}\mathrm{e}^{-\mathrm{i}F_z\tau}$, where (α,β,τ) are the Euler angles. Minimizing the energy with a fixed particle number, the ground state structure of $\Psi_a(\mathbf{r})$ can be found [5]. In the ferromagnetic state where $c_2 < 0$, the energy is minimized by $\langle\mathbf{F}\rangle^2 = 1$ and the ground state spinor and density are [5]

$$\zeta = \mathrm{e}^{\mathrm{i}\theta}\mathcal{U}\begin{pmatrix}1\\0\\0\end{pmatrix} = \mathrm{e}^{\mathrm{i}(\theta-\tau)}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\alpha}\cos^2\frac{\beta}{2}\\ \sqrt{2}\cos\frac{\beta}{2}\sin\frac{\beta}{2}\\ \mathrm{e}^{\mathrm{i}\alpha}\sin^2\frac{\beta}{2}\end{pmatrix}, \quad n^o(\mathbf{r}) = \frac{1}{c_0+c_2}\mu. \quad (2)$$

Because the distinct configurations of ζ in equation (2) are given by the full range of the Euler angles, the symmetry group of the ferromagnetic state is $SO(3)$ [5]. Introducing new variables $\vec{\mathbf{s}} = (s^1, s^2, s^3) = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$ and $\mathbf{C} = \frac{M}{en}\mathbf{J}$, where $\mathbf{J} = \frac{i\hbar en}{M}(\zeta^\dagger\nabla\zeta - \nabla\zeta^\dagger\zeta) - \frac{4e^2n}{Mc}\mathbf{A}$ is the supercurrent, the free energy (1) in the ferromagnetic state can be expressed as [2]

$$F = \int \mathrm{d}\mathbf{x}\left[\frac{\hbar^2}{2M}(\nabla\sqrt{n})^2 + \frac{\hbar^2 n}{4M}(\nabla\vec{\mathbf{s}})^2 + \frac{n}{8M}\mathbf{C}^2 + \frac{\hbar^2c^2}{128\pi e^2}\left(\epsilon_{abc}s_a\nabla s_b\times\nabla s_c - \frac{1}{\hbar}\nabla\times\mathbf{C}\right)^2 - \mu n + \frac{n^2}{2}\left[c_0+c_2\right]\right]. \quad (3)$$

Here and thereafter, summations over the repeated indices are assumed. According to free energy density (3), the magnetic field in the ferromagnetic spin-triplet superconductor is separated into two parts: the contribution from the supercurrent

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J and the self-induced magnetic field $\tilde{\mathbf{B}} \equiv \frac{1}{4e}\epsilon_{abc}s_a\nabla s_b \times \nabla s_c$, which is due to the nontrivial electromagnetic interaction between the components of ζ.

In [2], Babaev only considered the superconductor in a simply-connected space, which means the defects in the superconductor do not feature the zeros of the order parameter. Now, using the ϕ-mapping method, we will investigate the case with $n = 0$ at some isolated points, and find that these points correspond to magnetic monopoles. Furthermore, we will study possible world-line configurations of monopole–antimonopole pairs and multi-monopoles. Since $\pi_2(SO(3)) = 0$, these monopoles are topological unstable. Technically, they are the saddle points of the free energy (1), and can deform into the spin textures in principle. However, if we confine the parameters in the free energy (1) to a certain range, the relative stability of the monopoles can, to a large extent, be guaranteed. For simplicity, we will restrict our study to such a parameter range in this paper.

We begin by introducing an internal vector field

$$\vec{\mathbf{S}} = (S^1, S^2, S^3) = \sqrt{n}\vec{\mathbf{s}}, \tag{4}$$

and define a topological current as follows:

$$j^\mu = \frac{1}{4e}\epsilon^{\mu\nu\lambda\rho}\epsilon^{abc}\partial_\nu s^a\partial_\lambda s^b\partial_\rho s^c. \tag{5}$$

It is easy to see that j^μ is identically conserved. The corresponding conserved charge density $\rho = j^0 = \nabla\cdot\tilde{\mathbf{B}}$ serves precisely as the source of $\tilde{\mathbf{B}}$. Therefore we can speculate that j^μ is the current of the magnetic monopole. Using $\partial_\mu s^a = \partial_\mu S^a/|\vec{\mathbf{S}}| + S^a\partial_\mu(1/|\vec{\mathbf{S}}|)$, we obtain

$$j^\mu = -\frac{1}{4e}\epsilon^{\mu\nu\lambda\rho}\epsilon^{abc}\partial_d\partial_a\left(\frac{1}{|\vec{\mathbf{S}}|}\right)\partial_\nu S^d\partial_\lambda S^b\partial_\rho S^c, \tag{6}$$

where $\partial_a = \frac{\partial}{\partial S^a}$. Introducing the Jacobian vector

$$D^\mu\left(\frac{S}{x}\right) = \frac{1}{3!}\epsilon^{\mu\nu\lambda\rho}\epsilon^{abc}\partial_\nu S^a\partial_\lambda S^b\partial_\rho S^c, \tag{7}$$

and making use of the Green function relation in $\vec{\mathbf{S}}$-space:

$$\partial_a\partial_a\left(\frac{1}{|\vec{\mathbf{S}}|}\right) = -4\pi\delta(\vec{\mathbf{S}}), \tag{8}$$

we arrive at the following compact expression:

$$j^\mu = \frac{2\pi}{e}\delta(\vec{\mathbf{S}})D^\mu\left(\frac{S}{x}\right). \tag{9}$$

The δ-function included in equation (9) implies that j^μ can be nonzero only if $\vec{\mathbf{S}} = 0$. So the zero points of $\vec{\mathbf{S}}$ are important in determining the nontrivial j^μ. We assume $\vec{\mathbf{S}}$ has N isolated zero points denoted by $\mathbf{z}_r$ $(r = 1, \ldots, N)$. According to the implicit function theorem [6], $\vec{\mathbf{S}} = 0$ has an unique continuous solution under the regular condition

$$D^0\left(\frac{S}{x}\right)\Big|_{(\mathbf{z}_r,t)} \neq 0. \tag{10}$$

This solution can be expressed as

$$\mathbf{x}_r = \mathbf{z}_r(t), \tag{11}$$

which represents the N world lines of the magnetic monopoles. To further illustrate the topological and physical meaning of j^μ, we need a more detailed expression for $\delta(\vec{\mathbf{S}})$. In δ-function theory [7], given the regular condition (10), we can expand $\delta(\vec{\mathbf{S}})$ as follows:

$$\delta(\vec{\mathbf{S}}) = \sum_{r=1}^{N}\frac{W_r}{D^0(\frac{S}{x})|_{(\mathbf{z}_r,t)}}\delta(\mathbf{x} - \mathbf{z}_r(t)), \tag{12}$$

where W_r is the winding number of the ϕ-mapping. From the definitions of the Jacobian vector, we can obtain the velocity vector of the magnetic monopoles:

$$\frac{\mathrm{d}z_r^i(t)}{\mathrm{d}t} = \frac{D^i(S/x)}{D^0(S/x)}\Big|_{(\mathbf{z}_r(t),t)}. \tag{13}$$

From equations (9), (12) and (13), we find

$$j^i = \frac{2\pi}{e}\sum_{r=1}^{N}W_r\frac{\mathrm{d}z_r^i(t)}{\mathrm{d}t}\delta(\mathbf{x} - \mathbf{z}_r(t)), \tag{14}$$

$$\rho = j^0 = \frac{2\pi}{e}\sum_{r=1}^{N}W_r\delta(\mathbf{x} - \mathbf{z}_r(t)). \tag{15}$$

From equations (14) and (15), we can see that j^μ is indeed the current of the magnetic monopole. The nonvanishing of j^μ indicates the existence of the magnetic monopole. The corresponding magnetic charge of the rth monopole is given by the topological charge $\frac{2\pi}{e}W_r$. To make the energy finite in an infinite volume ferromagnetic spin-triplet superconductor, the magnetic monopoles can exist only in the form of the monopole–antimonopole pairs. In such a pair, the monopole and antimonopole will be connected by a Dirac string, or a doubly-quantized vortex, which belongs to the trivial topological class of $\pi_1(SO(3)) = Z_2$.

From the perspective of mathematics, we describe the monopoles by the vector field $\vec{\mathbf{S}}$, which does not include the Euler angle τ. This description cannot indicate the instability of the monopoles. A more detailed description must involve τ, and hence will reveal more detailed properties of the monopoles. We leave this subject to future studies.

When condition (10) fails at some fixed spacetime points $(\mathbf{z}_{r0}, t_0)$, i.e.

$$D^0\left(\frac{S}{x}\right)\Big|_{(\mathbf{z}_{r0},t_0)} = 0, \tag{16}$$

we call $(\mathbf{z}_{r0}, t_0)$ a branch point of the ϕ-mapping, or a branch point for short. In other words, the branch point is the point determined by $\vec{\mathbf{S}} = 0$ and $D^0(\frac{S}{x}) = 0$. There are two kinds of branch points, namely the limit points and the bifurcation points. If for at least one space index i,

$$D^i\left(\frac{S}{x}\right)\Big|_{(\mathbf{z}_{r0},t_0)} \neq 0, \tag{17}$$

we call $(\mathbf{z}_{r0}, t_0)$ a limit point. If for all space indices i,

$$D^i\left(\frac{S}{x}\right)\bigg|_{(\mathbf{z}_{r0},t_0)} = 0, \tag{18}$$

we call $(\mathbf{z}_{r0}, t_0)$ a bifurcation point.

Here we first discuss the evolution processes of the magnetic monopoles in the neighborhood of the limit point. Because of equation (16), we cannot use the implicit function theorem at the limit point as above. But we can use $D^1(\frac{S}{x})$ instead of $D^0(\frac{S}{x})$ to continue the discussion if we choose $i = 1$ in equation (17). This means that at the limit point $(\mathbf{z}_{r0}, t_0)$, $\vec{\mathbf{S}} = 0$ has an unique continuous solution, which can be expressed as

$$t = t(x^1), \tag{19}$$

$$x^2 = x^2(x^1), \tag{20}$$

$$x^3 = x^3(x^1). \tag{21}$$

Similar to equation (13), here we find

$$\frac{\mathrm{d}t}{\mathrm{d}x^1}\bigg|_{z^1_{r0}} = \frac{D^0(S/x)}{D^1(S/x)}\bigg|_{(\mathbf{z}_{r0},t_0)} = 0. \tag{22}$$

Then the Taylor expansion of equation (19) in the neighborhood of the limit point $(\mathbf{z}_{r0}, t_0)$ reads

$$t - t_0 = \frac{1}{2}\frac{\mathrm{d}^2 t}{(\mathrm{d}x^1)^2}\bigg|_{z^1_{r0}}(x^1 - z^1_{r0})^2 + (\text{higher order terms}). \tag{23}$$

Ignoring the higher order terms, equation (23) represents a parabola in the $x^1 - t$ plane. If $\frac{\mathrm{d}^2 t}{(\mathrm{d}x^1)^2}|_{(\mathbf{z}_{r0},t_0)} > 0$ (<0), this parabola implies that at the limit point $(\mathbf{z}_{r0}, t_0)$, there is a monopole–antimonopole pair created (annihilated). So we can conclude that the limit points of the ϕ-mapping are the points where the monopole–antimonopole pairs are created or annihilated.

Now we turn to the evolution processes of the magnetic monopoles in the neighborhood of the bifurcation point. Due to equations (16) and (18), the velocity vector of the magnetic monopole at the bifurcation point $(\mathbf{z}_{r0}, t_0)$ cannot be determined by equation (13). In general, there will be more than one velocity vector at the bifurcation point. So, in order to find out all velocity vectors at the bifurcation point $(\mathbf{z}_{r0}, t_0)$, we assume that

$$\left(\frac{\partial S^1}{\partial x^2}\frac{\partial S^2}{\partial x^3} - \frac{\partial S^1}{\partial x^3}\frac{\partial S^2}{\partial x^2}\right)\bigg|_{(\mathbf{z}_{r0},t_0)} \neq 0. \tag{24}$$

Then, from the implicit function theorem [6], it follows that there is an unique continuous solution to

$$S^1(\mathbf{x}, t) = 0, \qquad S^2(\mathbf{x}, t) = 0. \tag{25}$$

This solution can be expressed as

$$x^2 = f^2(x^1, t), \qquad x^3 = f^3(x^1, t). \tag{26}$$

Substituting equation (26) into (25), we obtain

$$S^1(x^1, f^2(x^1, t), f^3(x^1, t), t) \equiv 0, \tag{27}$$

$$S^2(x^1, f^2(x^1, t), f^3(x^1, t), t) \equiv 0. \tag{28}$$

From the differentiation of equations (27) and (28), and the Gaussian elimination method, we can find all the first and the second partial derivatives of $f^1(x^1, t)$ and $f^2(x^1, t)$. Substituting equation (26) into $S^3(\mathbf{x}, t) = 0$, we obtain

$$F(x^1, t) \equiv S^3(x^1, f^2(x^1, t), f^3(x^1, t), t) = 0. \tag{29}$$

From the definition of the branch point, we have

$$F(z^1_{r0}, t_0) = 0. \tag{30}$$

Using the partial derivatives of f^2 and f^3, as well as the Cramer rule, it can be proved that

$$\frac{\partial F}{\partial x^1}\bigg|_{(z^1_{r0},t_0)} = 0, \qquad \frac{\partial F}{\partial t}\bigg|_{(z^1_{r0},t_0)} = 0. \tag{31}$$

Then the Taylor expansion of equation (29) in the neighborhood of the bifurcation point $(\mathbf{z}_{r0}, t_0)$ reads

$$F(x^1, t) = \tfrac{1}{2}A(x^1 - z^1_{r0})^2 + B(x^1 - z^1_{r0})(t - t_0) + \tfrac{1}{2}C(t - t_0)^2 + (\text{higher order terms}) = 0, \tag{32}$$

where $A = \frac{\partial^2 F}{(\partial x^1)^2}|_{(z^1_{r0},t_0)}$, $B = \frac{\partial^2 F}{\partial x^1 \partial t}|_{(z^1_{r0},t_0)}$ and $C = \frac{\partial^2 F}{(\partial t)^2}|_{(z^1_{r0},t_0)}$. From the partial derivatives of f^2 and f^3, the constants A, B and C are calculable. Dividing equation (32) by $(t - t_0)^2$, and taking the limit $x^1 \to z^1_{r0}$ and $t \to t_0$, equation (29) gives rise to

$$A\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + 2B\frac{\mathrm{d}x^1}{\mathrm{d}t} + C = 0. \tag{33}$$

Similarly, we can also find

$$C\left(\frac{\mathrm{d}t}{\mathrm{d}x^1}\right)^2 + 2B\frac{\mathrm{d}t}{\mathrm{d}x^1} + A = 0. \tag{34}$$

From equation (33) or equation (34), we can obtain the velocity component $\frac{\mathrm{d}x^1}{\mathrm{d}t}$. The other velocity components $\frac{\mathrm{d}x^2}{\mathrm{d}t}$ and $\frac{\mathrm{d}x^3}{\mathrm{d}t}$ can be obtained from $\frac{\mathrm{d}x^1}{\mathrm{d}t}$ and the partial derivatives of f^2 and f^3. Therefore, the different velocity vectors of the magnetic monopoles at the bifurcation point $(\mathbf{z}_{r0}, t_0)$ can be determined completely.

According to the different values of A, B and C, there are four possible cases:

Case 1. For $A \neq 0$ and $B^2 - AC > 0$, we have two different solutions to equation (33). This case implies that two monopoles meet and then depart from each other at the bifurcation point.

Case 2. For $A \neq 0$ and $B^2 - AC = 0$, we have only one solution to equation (33). This case implies three different evolution processes: (a) one multi-monopole splits into two, (b) two monopoles merge into one, and (c) two monopoles tangentially intersect at the bifurcation point.

Case 3. For $A = 0$, $B \neq 0$ and $C \neq 0$, the velocity component $\frac{\mathrm{d}x^1}{\mathrm{d}t} = -\frac{C}{2B}$ or tends to infinity. This case implies two different evolution processes: (a) one multi-monopole splits into three, and (b) three monopoles merge into one at the bifurcation point.

Case 4. For $A = 0$ and $C = 0$, the velocity component $\frac{dx^1}{dt} = 0$ or tends to infinity. This case also implies two different evolution processes which are similar to case 3.

When condition (24) fails, and all other 2×2 sub-Jacobian also vanish at the bifurcation point $(\mathbf{z}_{r0}, t_0)$, we need to discuss the evolution processes at the higher order bifurcation point. This discussion will be more complicated, but the method will be similar to that we used above. From the above analysis, we can conclude that the bifurcation points of the ϕ-mapping are the points where the monopoles interact with each other.

In conclusion, using the ϕ-mapping method, we have argued that ferromagnetic spin-triplet superconductors allow the formation of unstable magnetic monopoles. The limit points and the bifurcation points of the ϕ-mapping serve as the interaction points of these magnetic monopoles.

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